HW V

$A \subseteq \mathbb{R}$, $c \in A^c$, $l, l_1, l_2 \in \mathbb{R}$. $f, g, f_i : A \to \mathbb{R}$.
(unless explicitly mentioned otherwise) $(i = 1, 2)$.

1. Suppose $f = g$ on $V_\delta(c) \cap (A \setminus \{c\})$ for some $\delta > 0$. Show that $\lim_{x \to c} f(x) = l$ iff $\lim_{x \to c} g(x) = l$.
2. Show, by def, that $\lim_{x \to c} f(x) = l$ iff $\lim_{x \to c} (f(x) - l) = 0$
3. Show, by def, that $\lim_{x \to c} f(x) = l$ iff $\lim_{x \to 0} f(x + c) = l$
4. For $A = \mathbb{R}$, show, by definition, that $\lim_{x \to 0} f(x) = l$ iff $\lim_{x \to 0} f(100x) = l$
5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 3 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Show that $\lim_{x \to c} f(x)$ exists in $\mathbb{R}$ iff $c = 3$.
6. Suppose $\lim_{x \to c} (f(x))^2 = l \geq 0$. Can we conclude that $\lim_{x \to c} f(x) = \sqrt{l}$? (Yes if $l = 0$ but not otherwise — counter-example ?)

7.* Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x + 2 & \text{if } x \in \mathbb{Q} \\ 3x - 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$
Exactly at what c such that $\lim_{x \to c} f(x)$ exists ?
And what is the limit then ?

8.* Find $\delta_i > 0$ such that on $V_{\delta_i}(1)$

$$|x^2 - 1| < \varepsilon_i$$

where $\varepsilon_1 = \frac{1}{2}$, $\varepsilon_2 = \frac{1}{10}$ and $\varepsilon_3 = \frac{1}{100}$

9.* Show that $x \in A^c$ iff $0 = \text{dist}(x, A \setminus \{x\}) := \inf\{|a - x| : a \in A \setminus \{x\}\}$

10.* Let $A = [0, \sqrt{2}) \cap \mathbb{Q}$ and let $f(x) = \text{dist}(x, A \setminus \{x\})$ $\forall x \in \mathbb{R}$.
express $f(x)$ explicitly in the form $f(x) = \begin{cases} \cdots \\ \cdots \\ \cdots \end{cases}$ and so determine A^c.

11.* Find $\delta > 0$ such that 2 is of (strictly) positive distance to the δ-neighbourhood $V_\delta(3)$ of 3.
Why $\delta = 1$ cannot do the job?
Show that $\lim_{x \to 3} \frac{x^2+1}{x-2} = 10$.

12. Let $\lim_{x \to c} g(x) = l_2 \neq 0$. Apply the def of limits to a suitable $\varepsilon > 0$ for getting $\delta > 0$, $k > 0$ such that $|g(x)| \geq k$ $\forall x \in V_\delta(c) \cap (A \setminus \{c\})$.
Why $\varepsilon = |l_2| > 0$ cannot do the job?
(Hopefully, this question can help you to understand our proof for the quotient rule).